

A Work Project, presented as part of the requirements for the Award of a  
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On the pricing of bivariate options in the presence of a discrete dividend  
payment

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**Topic of the work project:**

On the pricing of bivariate options in the presence of a discrete dividend payment

**Executive Summary:**

Under the assumptions of the Black & Scholes economy, I derive a pricing formula for European bivariate options where one of the underlyings pays a discrete dividend. While the price can be approximated to any precision, this is computationally costly. Notions of the extension of the approach to a higher number of underlyings are given.

**Key words**

bivariate option; discrete dividend; heat equation in finance

## 1. Purpose and structure of the work project

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The theory of valuation of options, which reportedly exist since centuries, and with it the research regarding the valuation of other derivatives took off with the publication of the famous Black & Scholes (1973) paper “The Pricing of Options and Corporate Liabilities”. With the use of a replicating portfolio they were able to eliminate the unpredictable component in the valuation formula and provide an analytic solution for the option price. The intuition and mechanics of their work are still the basis of the derivatives industry and of various subsequent threads of research which develop the pricing of derivatives beyond the somewhat restrictive assumptions of the Black & Scholes model.

One of these threads covers the valuation of derivatives written on several assets. Early papers include Margrabe’s (1978) valuation for the option to exchange one asset for another and Stulz’s (1982) expansion to different forms of options on two assets. Again, with more research the analytic formulas they provide were replaced by more sophisticated models, which especially refine the dependence structure between the assets. The advent of copula theory in finance in the late 1990s makes important contributions to this topic (Bouyé, 2000). The application possibilities for derivatives on multiple underlyings are manifold, like for example valuating rainbow options and bonds in foreign currencies.

Given that the Stulz valuation formula is based in the restrictive Black & Scholes economy, relaxing the assumptions will provide a better understanding of the model’s implications. In this work project, I examine the possibility to value a European call option on the maximum of two underlying stocks, where one pays a discrete dividend during the life of the option. I follow the approach introduced by Amaro de Matos et al. (2009): They find the exact value of an option on a single underlying paying a discrete dividend using a quasi-analytic formula by making use of the convexity properties of the solution to the option’s partial differential equation. In my case, the final formula implies the need to integrate numerically, which turns it into a pseudo-quasi-analytic formula. It yields the same precision benefit as in the one asset case, but without the feature of being quasi-analytic.

This work project is structured as follows: In the second section, I present the main findings of Amaro de Matos et al. (2009) on incorporating discrete dividend payments in the pricing formula for options and the relevant literature regarding the pricing formula for bivariate options. The third section presents the transformation of the initial partial differential equation into the heat equation which allows for the derivation of the pricing formula in section four. Section five deals with topics such as the implementation of the model in Matlab and the possibility to extend the results to more assets. Before concluding, section six briefly outlines the limitations to the findings of this work project.

## 2. Literature Review

Black & Scholes (1973) provide an analytic solution for the valuation of a European option without the need to estimate some kind of risk adjusted discount factor or to impose a utility function on the investors. The theoretical formula is only valid under what they call “ideal conditions” (p. 640), one of which is the absence of dividend payments or other distributions. This assumption is relaxed through the incorporation of a constant dividend yield into the model (Merton (1973)) or approximated by using the stock price reduced by the discounted discrete dividend (Black (1975)).

Amaro de Matos et al. (2009) present an approach for the valuation of a European call option on an underlying that pays a discrete dividend which is more precise than the Black approximation. They start with the Black & Scholes partial differential equation, transform it into the heat equation and receive an integral representation of the option value when introducing the initial condition/payoff function. Given that the discrete payment of a dividend at an intermediate point in time  $\tau$  introduces a jump into the process of the underlying, this process is not a geometric Brownian motion over the full life of the option anymore, but a piecewise geometric Brownian motion from  $t = 0$  until  $t = \tau$  and from  $t = \tau$  until maturity  $t = T$ . However, if the amount and point in time of the dividend payment is known in advance, the option price just before and just after the dividend payment will be the same, thus linking the two processes. With the new initial condition not being easily integrable, a different approach to the valuation is taken based on the convexity of the solution of the partial differential equation given a convex initial condition. An upper and lower bound of the option price can be constructed and the option price at  $t = 0$  can be priced with accuracy by choosing an arbitrarily small step-length between the knots of the upper-/lower-bound functions. The bounds are chords of piecewise linear functions and therefore easily integrable.

A simple extension to the plain vanilla option on one underlying is the option on two or more different underlyings, which can take on many forms depending on the final payoff function. In general, these options are subsumed under the name rainbow options. Margrabe (1978) is the first to analyze a bivariate option in the Black & Scholes framework for the case of the option to exchange one asset for another. His analytic solution is a special case of Stulz’s (1982) more general valuation formula for the bivariate case, which prices European puts and calls on the minimum or maximum of two different risky assets. This model relies on the constant correlation coefficient  $\rho$  as the measure of dependence between the assets. Ouwehand & West (2006) derive the same results as Stulz by applying the change-of-numeraire methodology, while also extending the results to any number of different underlyings.

Stulz (1982) derives his model for the valuation of European call options on the minimum/maximum of two assets in the Black & Scholes world. That is, the usual assumptions of a constant risk-free interest rate  $r$ , no dividend payments during the life of the option, no market frictions in form of transaction costs or taxes as well as continuous trading and unrestricted short-selling are made (Black & Scholes (1973)).

Also, assets are arbitrarily divisible and borrowing and lending is done at the same interest rate.

Furthermore, for the processes of the underlying securities A and B, geometric Brownian motions with correlated Wiener increments  $dW_A, dW_B$  are assumed where the coefficient of correlation  $\rho$  is constant.

$$dA = \mu_A A dt + \sigma_A A dW_A \quad (1.)$$

$$dB = \mu_B B dt + \sigma_B B dW_B \quad (2.)$$

with drift terms  $\mu_A, \mu_B$  and constant volatilities  $\sigma_A, \sigma_B$ .

Applying Itô's formula, the dynamics of a portfolio Z composed of the two underlyings can be derived:

$$dZ = \frac{\partial Z}{\partial A} dA + \frac{\partial Z}{\partial B} dB + \left( \frac{\partial Z}{\partial t} + \frac{1}{2} \frac{\partial^2 Z}{\partial A^2} \sigma_A^2 A^2 + \frac{\partial^2 Z}{\partial A \partial B} \rho \sigma_A \sigma_B AB + \frac{1}{2} \frac{\partial^2 Z}{\partial B^2} \sigma_B^2 B^2 \right) dt \quad (3.)$$

Also, the portfolio dynamics can be represented as the sum of its component dynamics, where  $r$  is the risk free interest rate:

$$dZ = aZ \frac{dA}{A} + bZ \frac{dB}{B} + (1 - a - b)rZdt \quad (4.)$$

Defining the weights of the risky assets A and B as

$$a = \frac{\partial Z}{\partial A} \frac{A}{Z} \quad (5.)$$

$$b = \frac{\partial Z}{\partial B} \frac{B}{Z} \quad (6.)$$

and setting (3.) equal to (4.) results in the partial differential equation (7.) every derivative on the two underlyings has to fulfill to exclude the possibility of arbitrage.

$$\frac{\partial Z}{\partial t} = rZ - \frac{\partial Z}{\partial A} Ar - \frac{\partial Z}{\partial B} Br - \frac{1}{2} \frac{\partial^2 Z}{\partial A^2} \sigma_A^2 A^2 - \frac{\partial^2 Z}{\partial A \partial B} \rho \sigma_A \sigma_B AB - \frac{1}{2} \frac{\partial^2 Z}{\partial B^2} \sigma_B^2 B^2 \quad (7.)$$

PDE

For a European call option on the minimum of the two assets  $Z_{\min}^{\text{Call}}(A, B, K, t)$ , also called call-on-min or worst of call, this equation has to be solved respecting the boundary condition

$$Z_{\min}^{\text{Call}}(A, B, K, 0) = \max [\min(A, B) - K; 0] \quad (8.)$$

where  $K$  is the strike price.

For this purpose, Stulz derives the bivariate density function  $f(\cdot)$  for the minimum  $k$  of the two assets and solves the integral

$$E[Z_{\min}^{\text{Call}}(A, B, K, 0)] = \int_{\ln K}^{\infty} e^{kf(k)} dk - \int_{\ln K}^{\infty} Kf(k) dk \quad (9.)$$

to receive the analytic formula for the option price  $Z_{\min}^{\text{Call}}(A, B, K, t)$ .

$$Z_{\min}^{\text{Call}}(A, B, K, t) = AN_2 \left( \gamma_1 + \sigma_A \sqrt{T-t}; \frac{\ln \frac{B}{A} - \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}; \frac{(\rho \sigma_B - \sigma_A)}{\sigma} \right) \quad (10.)$$

$$+ BN_2 \left( \gamma_2 + \sigma_B \sqrt{T-t}; \frac{\ln \frac{A}{B} - \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}; \frac{(\rho \sigma_A - \sigma_B)}{\sigma} \right)$$

$$- Ke^{-r(T-t)} N_2(\gamma_1; \gamma_2; \rho) \quad (11.)$$

$$\gamma_1 = \frac{\ln \frac{A}{K} + \left(r - \frac{1}{2} \sigma_A^2\right) (T-t)}{\sigma_A \sqrt{T-t}} \quad (12.)$$

$$\gamma_2 = \frac{\ln \frac{B}{K} + \left(r - \frac{1}{2} \sigma_B^2\right) (T-t)}{\sigma_B \sqrt{T-t}} \quad (13.)$$

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2\rho \sigma_A \sigma_B$$

where  $N_2$  is the bivariate standard normal cumulative distribution function with the upper integration limits in the first two arguments and the coefficient of correlation in the third.

The counterpart to the call option on the minimum of two assets is the call option on the maximum of two assets. Its price  $Z_{\max}^{\text{Call}}(A, B, K, t)$  is given by

$$Z_{\max}^{\text{Call}}(A, B, K, t) = C(A, K, t) + C(B, K, t) - Z_{\min}^{\text{Call}}(A, B, K, t) \quad (14.)$$

where  $C(A, K, t)$  and  $C(B, K, t)$  are the European call prices for the single underlyings with the same strike and time to expiration.

Ouwehand & West (2006) take a different approach to valuing the call-on-max and call-on-min: With the help of the change-of-numeraire methodology, they derive the analytic formula for these options for any number of underlyings. The idea behind this approach is to express the option value as the combined price of simpler derivatives that pay the value of, for example, the first asset if it has a higher value at maturity than the second asset. This first asset is also used as the numeraire, so that all prices can be expressed as ratios of it. The income stream from asset one serves as the risk-free rate, which is either a constant continuous dividend yield, or in the case where the asset does not pay dividends, zero. Additionally, the volatility of the price ratios has to be adapted. By adding up the prices for such derivatives for every underlying and subtracting the probability weighted discounted strike, the multivariate options are built. For a bivariate call-on-max on non-dividend-paying underlyings, the formula reads (for  $i, j, k \in \{A, B, K\}$ )

$$Z_{\max}^{\text{Call}}(A, B, K, t) = AN_2 \left( -d_{-}^{B/A}; d_{+}^{A}; \rho_{BK,A} \right) + B N_2 \left( -d_{-}^{A/B}; d_{+}^{B}; \rho_{AK,B} \right) \quad (15.)$$

$$- Ke^{-r(T-t)} [1 - N_2(-d_{-}^{A}; -d_{-}^{B}; \rho)] \quad (16.)$$

$$\sigma_{i/j}^2 = \sigma_i^2 + \sigma_j^2 - 2\rho \sigma_i \sigma_j$$

$$d_{-}^{i/j} = \frac{\ln\left(\frac{i}{j}\right) - \frac{1}{2}\sigma_{i/j}^2(T-t)}{\sigma_{i/j}\sqrt{T-t}} \quad (17.)$$

$$d_{\pm}^i = \frac{\ln\left(\frac{i}{K}\right) + \left(r \pm \frac{1}{2}\sigma_i^2\right)(T-t)}{\sigma_i\sqrt{T-t}} \quad (18.)$$

$$\rho_{ij,k} = \frac{\rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \sigma_k^2}{\sqrt{(\sigma_i^2 + \sigma_k^2 - 2\rho_{ik}\sigma_i\sigma_k)(\sigma_j^2 + \sigma_k^2 - 2\rho_{jk}\sigma_j\sigma_k)}} \quad (19.)$$

### 3. Transformation of the initial partial differential equation into the heat equation

The derivation of the pseudo-quasi-analytic formula for pricing the bivariate option follows the directions of Amaro de Matos et al. The first step is to translate the partial differential equation into the heat equation in  $\mathbb{R}^2$ . The general formula for the initial value problem of the heat equation of dimension  $n$  is

$$u_t = \kappa \Delta u \quad (20.)$$

$$u(x, 0) = g(x) \quad (21.)$$

with the vector  $x$  in  $\mathbb{R}^n$ ,  $t > 0$ ,  $\Delta$  the Laplace operator<sup>1</sup> and  $\kappa$  a constant. The solution for this partial differential equation and initial value (Levandosky, 2003) is

$$u(x, t) = \frac{1}{(4\kappa\pi t)^{n/2}} \int_{\mathbb{R}^n} g(y) \exp\left(-\frac{|x-y|^2}{4\kappa t}\right) dy \quad (22.)$$

where  $|x-y|^2$  is defined as  $\sum_{i=1}^n (x_i - y_i)^2$ .

For the purpose of the transformation, define

$$\theta = T - t \quad (23.)$$

$$x_1 = \frac{-\theta[2r\sigma_B + \rho\sigma_A\sigma_B^2 - 2r\rho\sigma_A - \sigma_A^2\sigma_B] - 2\sigma_B\log(A) + 2\rho\sigma_A\log(B)}{\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \quad (24.)$$

$$x_2 = \frac{-\theta(\sigma_B^2 - 2r) + 2\log(B)}{\sqrt{2}\sigma_B} \quad (25.)$$

$$\Phi(x, \theta) = e^{r(T-t)} Z(A, B, K, t) \quad (26.)$$

With these changes of variables, the PDE in  $\mathbb{R}^2$  simplifies to the heat equation with unitary constant  $\kappa$ .

$$\frac{\partial \Phi}{\partial \theta} = \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} \quad (27.)$$

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<sup>1</sup>  $\Delta f(x) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$

For the bivariate call-on-max, the translation of the boundary condition  $\max[\max(A; B) - K; 0]$  at maturity into the initial condition for the heat equation is also necessary:

$$\begin{aligned}\Phi(x, 0) &= Z_{\max}^{\text{Call}}(A, B, K, T) = \max[\max(A; B) - K; 0] \\ &= \max \left[ \max \left[ e^{\left(\rho x_2 - \sqrt{1-\rho^2} x_1\right) \frac{\sigma_A}{\sqrt{2}}}, e^{\frac{x_2 \sqrt{2} \sigma_B}{2}} \right] - K; 0 \right]\end{aligned}\quad (28.)$$

Defining the initial condition  $\Phi(x, 0)$  as  $g(\cdot)$ , the price of the bivariate call-on-max in integral form is

$$\begin{aligned}Z_{\max}^{\text{Call}}(A, B, K, t) &= e^{-r\theta} \Phi(x, \theta) \\ &= \frac{e^{-r\theta}}{4\pi\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y) \exp \left( -\frac{1}{2} \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2\theta} \right) dy_2 dy_1\end{aligned}\quad (29.)$$

Integrating this expression constitutes an alternative way to retrieve the analytic valuation formula of the bivariate call-on-max.

One useful property of the solution to the heat equation is that - given a convex, non-decreasing, non-negative initial condition - the solution is convex and non-decreasing as well, as demonstrated below.

#### Definition<sup>2</sup> of convexity in $\mathbb{R}^n$

A function  $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  is convex if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all  $x, y \in \mathbb{R}^n$ ,  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 1$ .

The general solution for the heat equation in  $\mathbb{R}^2$  is

$$u(x, t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} g(y) \exp \left( -\frac{|x - y|^2}{4t} \right) dy \quad (30.)$$

with initial condition

$$u(x, 0) = g(x) \quad (31.)$$

Set  $z = y - x$  and suppose that  $f(\cdot)$  is convex and non-negative. Then

$$\begin{aligned}f \left[ \alpha \begin{pmatrix} x_1^1 + z_1 \\ x_2^1 + z_2 \end{pmatrix} + \beta \begin{pmatrix} x_1^2 + z_1 \\ x_2^2 + z_2 \end{pmatrix} \right] &= f \left[ \alpha \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} + \beta \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} + z \right] \\ &\leq \alpha f \left[ \begin{pmatrix} x_1^1 + z_1 \\ x_2^1 + z_2 \end{pmatrix} \right] + \beta f \left[ \begin{pmatrix} x_1^2 + z_1 \\ x_2^2 + z_2 \end{pmatrix} \right]\end{aligned}\quad (32.)$$

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<sup>2</sup> Pendavingh (2006)



where

$$x_i = \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix} \quad (33.)$$

Inserting in the general solution, this yields

$$\begin{aligned} u\left(\alpha \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} + \beta \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, t\right) &= \frac{1}{4\pi t} \int_{\mathbb{R}^2} \exp\left(-\frac{|z|^2}{4t}\right) g\left[\alpha \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} + \beta \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} + z\right] dz \\ &\leq \frac{1}{4\pi t} \int_{\mathbb{R}^2} \exp\left(-\frac{|z|^2}{4t}\right) \left\{ \alpha g\left[z + \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}\right] + \beta g\left[z + \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}\right] \right\} dz \\ &= \alpha u\left(z + \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}, t\right) + \beta u\left(z + \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, t\right) \end{aligned} \quad (34.)$$

which states that the solution  $u(x, t)$  is convex. With  $g(x) = \max(\max(A, B) - K; 0)$ , a convex, non-negative and non-decreasing function in both  $A$  and  $B$ , the price of the bivariate call-on-max is convex and increasing. In contrast, if  $g(x) = \max(\min(A, B) - K; 0)$ , the initial condition is not convex, and therefore neither is the price of the bivariate call-on-min.

#### 4. Pseudo-quasi-analytic formula for pricing a bivariate option in the presence of a discrete dividend payment

Now assume that stock  $A$  pays a known discrete dividend  $D$  at time  $\tau$  during the life of the option. This introduces a jump into the price path of the stock. While stock  $B$  follows a geometric Brownian motion over the full life of the option, the price path of stock  $A$  is split into two different geometric Brownian motions at the time of the dividend payment. However, the derivative's price is not supposed to jump when the dividend is paid, as the date of payment and amount paid is known beforehand. Therefore, right before and right after the dividend payment, the option has the same value.

This means that at time  $\tau$ , for  $A^{\text{antes}}$  as the asset's price right before the dividend payment (equal to the last price of the first stochastic process of asset  $A$ ) and  $A^{\text{post}}$  as the asset's price right after the dividend payment,  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$  is equal to  $Z_{\max}^{\text{Call}}(A^{\text{post}}, B, K, \tau)$ . Under the assumption that  $A^{\text{antes}} - D > 0$  and inserting  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$  into the general solution of the heat equation in  $\mathbb{R}^2$  as the new initial condition gives

$$\begin{aligned} Z_{\max}^{\text{Call}}(A, B, K, \tau) &= e^{-r\theta} \Phi(x, \theta) \\ &= \frac{e^{-r\theta}}{4\pi\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau) \\ &\quad * \exp\left(-\frac{1}{2} \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2\theta}\right) dy_2 dy_1 \end{aligned} \quad (35.)$$

Note that  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$  is also a convex, non-decreasing, non-negative function, so the solution to this problem will inherit these characteristics. However, integrating this new expression is not easily possible. Instead, using the convexity property and the limiting behavior of the initial condition, an integrable upper and a lower bound of the initial condition will be derived and integrated.

For the construction of the bounds, it is important to understand the limiting behavior of  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$ . In the simplest case, either  $A^{\text{antes}}$  or  $B$  tend to infinity while the other is fixed to a much smaller value. Then, using the Ouwehand & West formula, it can easily be shown that the price tends to  $A^{\text{antes}} - D - Ke^{-r(\tau-t)}$  and  $B - Ke^{-r(\tau-t)}$  respectively. In this case, the derivative of the option price with respect to the asset price that tends to infinity is close to one. Now suppose  $A^{\text{antes}}$  tends to infinity but  $B$  has a non-negligible size as well. On the one hand, the derivative of the option price with respect to  $A^{\text{antes}}$  is a decreasing function in  $B$ . The inclination and limit of the derivative depend on the values of the other option price parameters. Importantly, the function only takes values between one and zero. Figure 1 qualitatively shows the behavior of the derivative for increasing  $B$ . On the other hand, the derivative of the option price with respect to  $B$  is an increasing function in  $B$  with limit below one. This is depicted in Figure 2. The behavior of the first derivatives of the option price when  $B$  tends to infinity and  $A^{\text{antes}}$  is growing follows the same lines.

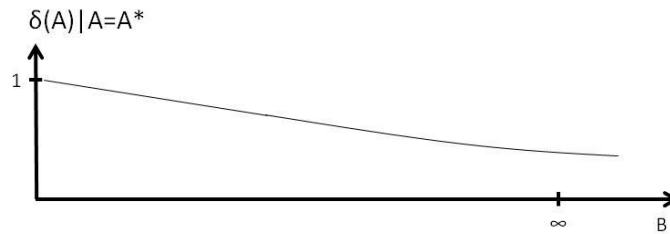


Figure 1: First partial derivative of  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$  with respect to  $A^{\text{antes}}$

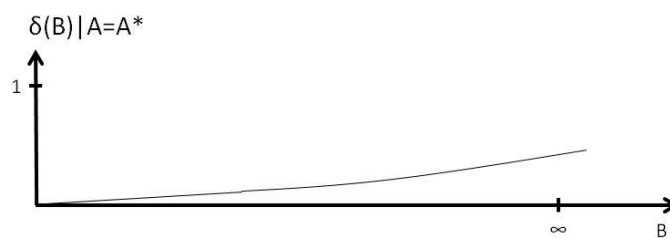


Figure 2: First partial derivative of  $Z_{\max}^{\text{Call}}(A^{\text{antes}} - D, B, K, \tau)$  with respect to  $B$

In combination with the convexity property of the bivariate option price, this means that by choosing appropriate inclinations for a surface in the  $(A, B)$ -space a surface which lies above as well as a surface which lies below the true option prices at time  $\tau$  can be constructed.

For the construction of the upper bound and the lower bound, the area spanned up by the range of possible prices of  $A$  and  $B$  will be split up in four subareas. This is achieved by choosing two prices  $A^*$  and  $B^*$  which are sufficiently large ( $A^*$  has to be at least

greater than the dividend  $D$ ) and serve as the points of separation. Figure 3 below depicts this partition.

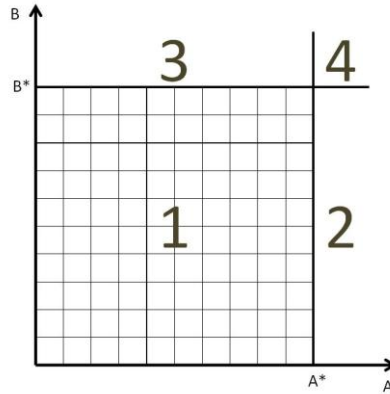


Figure 3: Separation of subareas and grid in subarea 1

For the upper bound, the first subarea is further divided into sections by splitting  $[D; A^* - D]$  into  $M_A$  parts and  $[0; B^*]$  into  $M_B$  parts, where  $M_A$  and  $M_B$  are integers bigger than one. The resulting points on the  $A$  and  $B$  axis are defined as follows:

$$A_i = D + i\Delta A \quad i = 1 \dots M_A \quad (36.)$$

$$B_j = j\Delta B \quad j = 1 \dots M_B \quad (37.)$$

$$\Delta A = \frac{A^* - D}{M_A} \quad (38.)$$

$$\Delta B = \frac{B^*}{M_B} \quad (39.)$$

For every intersection  $(A_i, B_j)$ , calculate the price of  $Z_{\max}^{\text{Call}}(A_i - D, B_j, K, \tau)$ . These prices span up a surface in the  $(A, B, Z_{\max}^{\text{Call}})$ -space which lies above the possible true prices of the bivariate option at time  $\tau$  due to the option price's convexity. Figure 4a gives an example for an arbitrary subsection of the surface between  $(A_{i-1}, B_{j-1})$  and  $(A_i, B_j)$ .

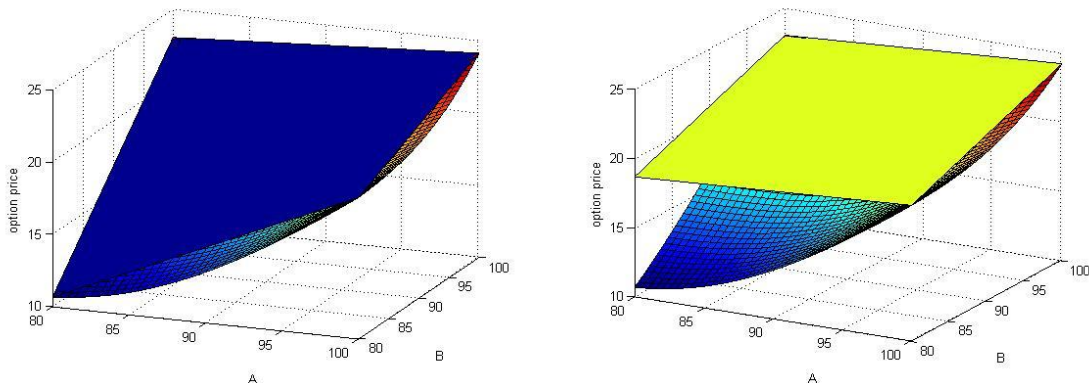


Figure 4a/b: Subsections of the true and approximated upper option price surfaces at time  $\tau$

However, to further facilitate the integration, define

$$\alpha_{i,j} = \frac{M_A}{A^* - D} [Z_{\max}^{\text{Call}}(A_i - D, B_j, K, \tau) - Z_{\max}^{\text{Call}}(A_{i-1} - D, B_j, K, \tau)] \quad (40.)$$

$$\beta_{i,j} = \frac{M_B}{B^*} [Z_{\max}^{\text{Call}}(A_i - D, B_j, K, \tau) - Z_{\max}^{\text{Call}}(A_i - D, B_{j-1}, K, \tau)] \quad (41.)$$

which yields the inclinations of the surface in A-direction for constant  $B_j$  and B-direction for constant  $A_i$ . Figure 4b depicts the new simplified surface for the same subsection as before.

The simplified option price surface in subarea one is then

$$Z_1^+ = \sum_{i=1}^{M_A} \sum_{j=1}^{M_B} [\alpha_{i,j}(A - A_i) + \beta_{i,j}(B - B_j) + Z_{\max}^{\text{Call}}(A_i - D, B_j, K, \tau)] * \chi_{[A_{i-1}, A_i]}[A] * \chi_{[B_{j-1}, B_j]}[B] \quad (42.)$$

where  $\chi_{[\Psi_{i-1}, \Psi_i]}[\Psi]$  serves as an indicator function which assumes the value 1 if  $\Psi \in [\Psi_{i-1}, \Psi_i]$  and 0 otherwise.

Next, for subareas two, three and four, based on the limiting behavior of the option price at time  $\tau$ , it is assumed that the product becomes delta-one in the underlying with price above  $A^*$  and  $B^*$  respectively. This approximation works very well in cases where  $A \gg B$  or  $B \gg A$ , i.e. at the axis margins of subareas two and three, but gets less and less precise towards subarea four. However, the imprecision does not impact the upper bound substantially as the probability mass in this subarea is very small.

Working in the same way as for subarea one, the simplified surfaces in the subareas two, three and four are:

$$Z_2^+ = \sum_{j=1}^{M_B} [(A - A^*) + \beta_{M_A, j}(B - B_j) + Z_{\max}^{\text{Call}}(A^* - D, B_j, K, \tau)] * \chi_{[A^*, \infty)}[A] * \chi_{[B_{j-1}, B_j]}[B] \quad (43.)$$

$$Z_3^+ = \sum_{i=1}^{M_A} [\alpha_{i, M_B}(A - A_i) + (B - B^*) + Z_{\max}^{\text{Call}}(A_i - D, B^*, K, \tau)] * \chi_{[A_{i-1}, A_i]}[A] * \chi_{[B^*, \infty)}[B] \quad (44.)$$

$$Z_4^+ = [(A - A^*) + (B - B^*) + Z_{\max}^{\text{Call}}(A^* - D, B^*, K, \tau)] * \chi_{[A^*, \infty)}[A] * \chi_{[B^*, \infty)}[B] \quad (45.)$$

The whole simplified surface for the upper bound is thus  $Z_{\text{surface}}^{\text{upper}} = Z_1^+ + Z_2^+ + Z_3^+ + Z_4^+$ .

The construction of the lower bound follows the same procedure with one adaption. The surface cannot be laid through neighboring option prices at time  $\tau$  anymore. Instead, making use of the convexity again, the price of the option at every midpoint  $(A_{i-0.5}, B_{j-0.5})$  of all subsections  $[A_{i-1}, A_i] * [B_{j-1}, B_j]$  as well as the partial derivatives<sup>3</sup> of the option price with respect to the asset prices will be used to lay a piecewise tangent surface below the true price surface. Figure 5 illustrates this procedure for one subsection.

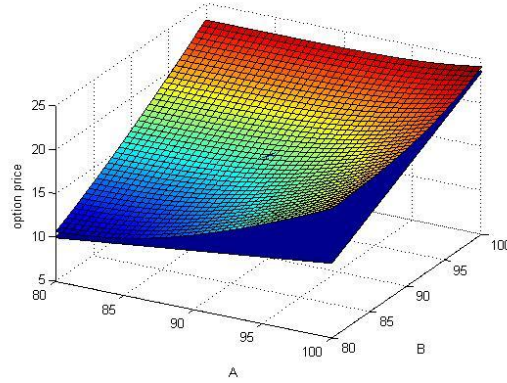


Figure 5: Subsection of the true and approximated lower option price surfaces at time  $\tau$

The lower bound surfaces for the different areas are defined by the following four expressions so that  $Z_{\text{surface}}^{\text{lower}} = Z_1^- + Z_2^- + Z_3^- + Z_4^-$ .

$$Z_1^- = \sum_{i=1}^{M_A} \sum_{j=1}^{M_B} [Z^{A'}(A_{i-0.5}, B_{j-0.5}, K, \tau)(A - A_{i-0.5}) + Z^{B'}(A_{i-0.5}, B_{j-0.5}, K, \tau)(B - B_{j-0.5}) + Z_{\text{max}}^{\text{Call}}(A_{i-0.5} - D, B_{j-0.5}, K, \tau)] * \chi_{[A_{i-1}, A_i]}[A] * \chi_{[B_{j-1}, B_j]}[B] \quad (46.)$$

$$Z_2^- = \sum_{j=1}^{M_B} [Z^{A'}(A^*, B_j, K, \tau)(A - A^*) + Z^{B'}(A^*, B_{j-1}, K, \tau)(B - B_{j-1}) + Z_{\text{max}}^{\text{Call}}(A^* - D, B_{j-1}, K, \tau)] * \chi_{[A^*, \infty)}[A] * \chi_{[B_{j-1}, B_j]}[B] \quad (47.)$$

$$Z_3^- = \sum_{i=1}^{M_A} [Z^{A'}(A_{i-1}, B^*, K, \tau)(A - A_{i-1}) + Z^{B'}(A_i, B^*, K, \tau)(B - B^*) + Z_{\text{max}}^{\text{Call}}(A_{i-1} - D, B^*, K, \tau)] * \chi_{[A_{i-1}, A_i]}[A] * \chi_{[B^*, \infty)}[B] \quad (48.)$$

$$Z_4^- = [Z^{A'}(A^*, B^*, K, \tau)(A - A^*) + Z^{B'}(A^*, B^*, K, \tau)(B - B^*) + Z_{\text{max}}^{\text{Call}}(A^*, B^*, K, \tau)] * \chi_{[A^*, \infty)}[A] * \chi_{[B^*, \infty)}[B] \quad (49.)$$

<sup>3</sup> See Annex for the formula of the partial derivatives  $Z^{A'}(A_{i-0.5}, B_{j-0.5}, K, \tau)$  and  $Z^{B'}(A_{i-0.5}, B_{j-0.5}, K, \tau)$ .

Inserting the upper and lower surface in the general solution to the heat equation and integrating yields the formulas for the upper bound  $Z_{\max}^{\text{Call,upper}}$  and lower bound  $Z_{\max}^{\text{Call,lower}}$  of the price of the bivariate call-on-max with one discrete dividend payment.

$$Z_{\max}^{\text{Call,upper}} = Z_1^{++} + Z_2^{++} + Z_3^{++} + Z_4^{++} \quad (50.)$$

$$Z_1^{++} = \sum_{i=1}^{M_A} \sum_{j=1}^{M_B} \left[ \alpha_{i,j} \text{AP} \left( f_1^i + \lambda_1 + \gamma_1 + \omega_2, f_2^j + \lambda_2 \right) + \beta_{i,j} \text{BP} \left( f_1^i + \gamma_2 + \omega_2, f_2^j + \lambda_4 \right) \right. \quad (51.)$$

$$\left. + \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A_i - D, B_j, K, \tau) - \alpha_{i,j} A_i - \beta_{i,j} B_j \right] P \left( f_1^i + \omega_2, f_2^j \right) \right]$$

$$Z_2^{++} = \sum_{j=1}^{M_B} \left[ \text{AP} \left( f_1^{M_A} + \lambda_1 + \gamma_1 + \omega_2, f_2^j + \lambda_2 \right) + \beta_{M_A,j} \text{BP} \left( f_1^{M_A} + \gamma_2 + \omega_2, f_2^j + \lambda_4 \right) \right. \quad (52.)$$

$$\left. + \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A^* - D, B_{j-1}, K, \tau) - \beta_{M_A,j} B_{j-1} - A^* \right] P \left( f_1^{M_A} + \omega_2, f_2^j \right) \right]$$

$$Z_3^{++} = \sum_{i=1}^{M_A} \left[ \alpha_{i,M_B} \text{AP} \left( f_1^i + \lambda_1 + \gamma_1 + \omega_2, f_2^{M_B} + \lambda_2 \right) + \text{BP} \left( f_1^i + \gamma_2 + \omega_2, f_2^{M_B} + \lambda_4 \right) \right. \quad (53.)$$

$$\left. + \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A_{i-1} - D, B^*, K, \tau) - \alpha_{i,M_B} A_{i-1} - B^* \right] P \left( f_1^i + \omega_2, f_2^{M_B} \right) \right]$$

$$Z_4^{++} = \text{AP} \left( f_1^{M_A} + \lambda_1 + \gamma_1 + \omega_2, f_2^{M_B} + \lambda_2 \right) + \text{BP} \left( f_1^{M_A} + \gamma_2 + \omega_2, f_2^{M_B} + \lambda_4 \right) \quad (54.)$$

$$+ \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A^* - D, B^*, K, \tau) - A^* - B^* \right] P \left( f_1^{M_A} + \omega_2, f_2^{M_B} \right)$$

$$Z_{\max}^{\text{Call,lower}} = Z_1^{--} + Z_2^{--} + Z_3^{--} + Z_4^{--} \quad (55.)$$

$$Z_1^{--} = \sum_{i=1}^{M_A} \sum_{j=1}^{M_B} \left[ Z^{A'}(A_{i-0.5}, B_{j-0.5}, K, \tau) \text{AP} \left( f_1^i + \lambda_1 + \gamma_1 + \omega_2, f_2^j + \lambda_2 \right) \right. \quad (56.)$$

$$\left. + Z^{B'}(A_{i-0.5}, B_{j-0.5}, K, \tau) \text{BP} \left( f_1^i + \gamma_2 + \omega_2, f_2^j + \lambda_4 \right) + \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A_{i-0.5} - D, B_{j-0.5}, K, \tau) - Z^{A'}(A_{i-0.5}, B_{j-0.5}, K, \tau) A_{i-0.5} - Z^{B'}(A_{i-0.5}, B_{j-0.5}, K, \tau) B_{j-0.5} \right] \right. \\ \left. * P \left( f_1^i + \omega_2, f_2^j \right) \right]$$

$$Z_2^{--} = \sum_{j=1}^{M_B} \left[ Z^{A'}(A^*, B_j, K, \tau) \text{AP} \left( f_1^{M_A} + \lambda_1 + \gamma_1 + \omega_2, f_2^j + \lambda_2 \right) \right. \quad (57.)$$

$$\left. + Z^{B'}(A^*, B_{j-1}, K, \tau) \text{BP} \left( f_1^{M_A} + \gamma_2 + \omega_2, f_2^j + \lambda_4 \right) + \exp(-r\theta_\tau) \left[ Z_{\max}^{\text{Call}}(A^* - D, B_{j-1}, K, \tau) - Z^{A'}(A^*, B_j, K, \tau) A^* - Z^{B'}(A^*, B_{j-1}, K, \tau) B_{j-1} \right] \right] P \left( f_1^{M_A} + \omega_2, f_2^j \right)$$

$$\begin{aligned}
Z_3^{--} = \sum_{i=1}^{M_A} & [Z^{A'}(A_{i-1}, B^*, K, \tau) AP(f_1^i + \lambda_1 + \gamma_1 + \omega_2, f_2^{M_B} + \lambda_2) \\
& + Z^{B'}(A_i, B^*, K, \tau) BP(f_1^i + \gamma_2 + \omega_2, f_2^{M_B} + \lambda_4) \\
& + \exp(-r\theta_\tau) [Z_{\max}^{\text{Call}}(A_{i-1} - D, B^*, K, \tau) - Z^{A'}(A_{i-1}, B^*, K, \tau) A_{i-1} \\
& - Z^{B'}(A_i, B^*, K, \tau) B^*] P(f_1^i + \omega_2, f_2^{M_B})]
\end{aligned} \tag{58.}$$

$$\begin{aligned}
Z_4^{--} = Z^{A'}(A^*, B^*, K, \tau) & AP(f_1^{M_A} + \lambda_1 + \gamma_1 + \omega_2, f_2^{M_B} + \lambda_2) \\
& + Z^{B'}(A^*, B^*, K, \tau) BP(f_1^{M_A} + \gamma_2 + \omega_2, f_2^{M_B} + \lambda_4) \\
& + \exp(-r\theta_\tau) [Z_{\max}^{\text{Call}}(A^* - D, B^*, K, \tau) - Z^{A'}(A^*, B^*, K, \tau) A^* \\
& - Z^{B'}(A^*, B^*, K, \tau) B^*] P(f_1^{M_A} + \omega_2, f_2^{M_B})
\end{aligned} \tag{59.}$$

where

$$\theta_\tau = \tau - t \tag{60.}$$

$$f_1^i = \frac{-\theta_\tau [2r\sigma_B - \sigma_A^2\sigma_B] - 2\sigma_B \log\left(\frac{A}{A_i}\right)}{2\sqrt{\theta_\tau}\sigma_A\sigma_B\sqrt{1-\rho^2}} \tag{61.}$$

$$f_2^j = \frac{-\theta_\tau(\sigma_B^2 - 2r) + 2\log\left(\frac{B}{B_j}\right)}{2\sqrt{\theta_\tau}\sigma_B} \tag{62.}$$

$$\lambda_1 = -\sigma_A\sqrt{1-\rho^2}\sqrt{\theta_\tau} \tag{63.}$$

$$\lambda_2 = \rho\sigma_A\sqrt{\theta_\tau} \tag{64.}$$

$$\lambda_4 = \sigma_B\sqrt{\theta_\tau} \tag{65.}$$

$$\gamma_1 = -\frac{\rho^2\sigma_A^2\sigma_B\theta_\tau}{\sqrt{\theta_\tau}\sigma_A\sigma_B\sqrt{1-\rho^2}} \tag{66.}$$

$$\gamma_2 = -\frac{\rho\sigma_A\sigma_B^2\theta_\tau}{\sqrt{\theta_\tau}\sigma_A\sigma_B\sqrt{1-\rho^2}} \tag{67.}$$

$$\omega_2 = \frac{\rho}{\sqrt{1-\rho^2}} f_2 \tag{68.}$$

$$P(f_1^i, f_2^j) = \frac{1}{2\pi} \int_{f_1^{i-1}}^{f_1^i} \int_{f_2^{j-1}}^{f_2^j} \exp\left(-\frac{1}{2}[(z_1)^2 + (z_2)^2]\right) dz_2 dz_1 \tag{69.}$$

The function  $P(f_1^i, f_2^j)$  and its first argument deserve a closer look: At first sight the function has the form of a bivariate standard normal distribution for which tabulated values exist. However, the factor  $\omega_2$  which appears in the first argument every time the function is called prevents a simple calculation.  $\omega_2$  contains the second limit of

integration, which means that the integral has to be solved numerically as no tabulated values exist. This intricacy makes the formula a pseudo-quasi-analytic one. Consequently, one of the main findings of the approach of the one-asset-case is lost when it is extended to more assets. In some special cases discussed later, the need for numerical integration does not apply; easier methods than the full formula can be used in these cases. Also, the calculation time for the option price is greatly increased, an issue the next section will treat in more detail.

## 5. Afterthoughts on the approach

### 5.1 On the implementation of the model in Matlab

While most of the implementation is straightforward, two issues are worth mentioning: The first one is the expansion of calculation time needed as soon as one wants to get closer upper and lower bounds by increasing the number of meshes in the grid in the first subarea. Increasing the values for  $M_A$  and  $M_B$  linearly raises the number of knots quadratically.

The second issue is the need to use numerical integration, which is both costly in terms of calculation time and leads to a problem at the edges of the  $(A, B)$ -space, i.e. when either  $A$  or  $B$  or both tend to infinity. Here, Matlab is unable to calculate the integral numerically, which prohibits the calculation of the three outer subareas. This problem is overcome by choosing large values for  $A^*$  and  $B^*$  so that the probability masses in the outer subareas are virtually zero. At the same time, the number of meshes in the first subarea have to be considerably raised to retain the same level of precision. One possible way of improving the code would be to construct the grid in a way that knots are denser in areas where the variation in probability mass is higher, and less dense where the inclinations of the probability distribution function do change only by a small amount from knot to knot. Figure 6 displays the set-up with both the standard calculation method and one improved calculation method with different knot distances. Table 1 and Table 2 show the results in terms of gains in precision and costs in calculation time. The parameter values are  $A = 75$ ,  $B = 55$ ,  $K = 60$ ,  $D = 5$ ,  $\rho = 0.7$ ,  $\sigma_A = 0.2$ ,  $\sigma_B = 0.3$ ,  $r = 0.05$ ,  $T = 1$ ,  $\tau = 0.5$ ,  $t = 0$ ,  $A^* = 200$ ,  $B^* = 165$  and the Black approximation of the option price is 14.8484 units of monetary currency.



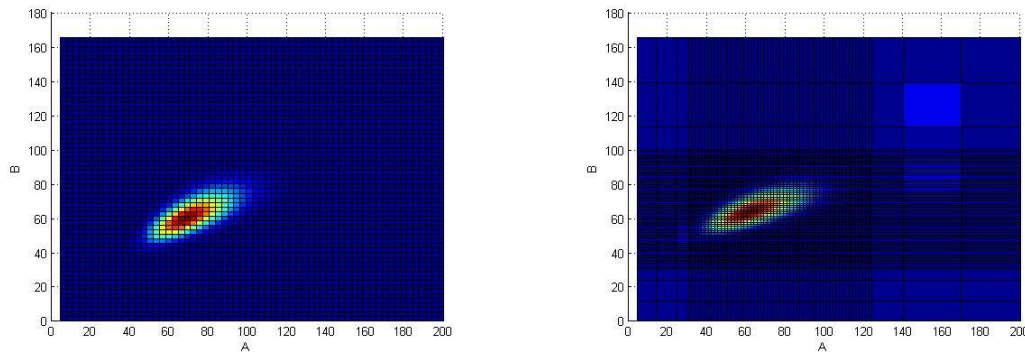


Figure 6: Knot density for standard and improved grid construction method

Knot density	Upper price limit	Lower price limit	Calculation time (in min) <sup>4</sup>
25	15.2777	14.8594	1.72
50	15.0309	14.9259	6.15
100	14.9693	14.9430	24.27
200	14.9540	14.9474	95.17
400	14.9501	14.9485	397.18

Table 1: Upper and lower price limits for different knot densities with the standard grid construction method

Knot density	Upper price limit	Lower price limit	Calculation time (in min)
25	15.1451	14.8923	1.72
50	14.9773	14.9406	5.98
75	14.9602	14.9455	13.12
100	14.9551	14.9470	22.12
125	14.9530	14.9476	33.57

Table 2: Upper and lower price limits for different knot densities with the improved grid construction method

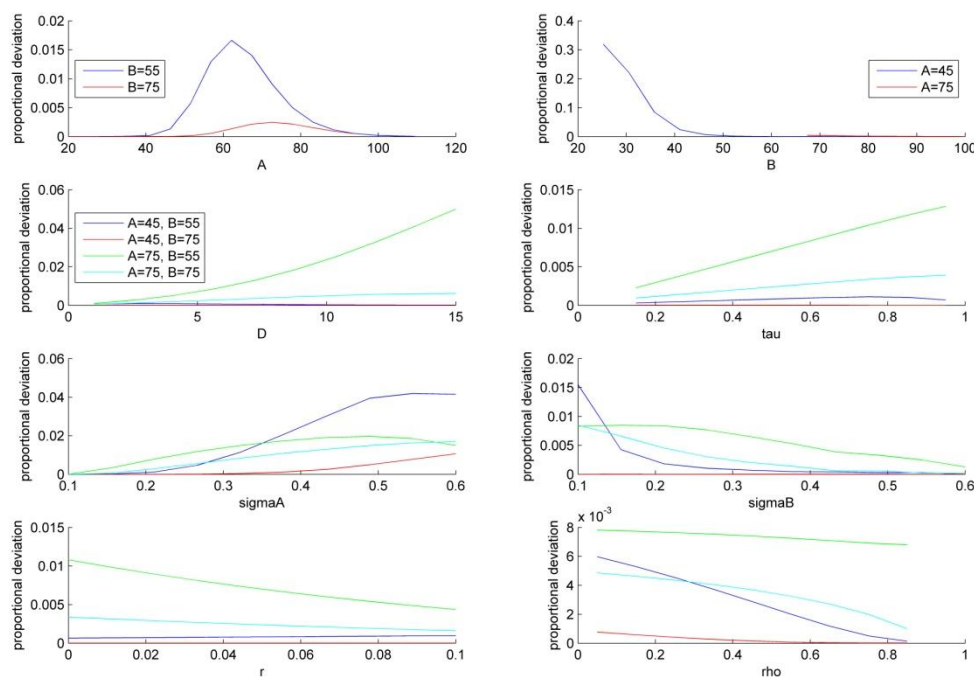
In the above example the Black approximation misprices the option by 0.10 monetary units at approximately 0.7% below the true value. The next section will take a closer look on the behavior of the difference between the Black approximation and the true option price for different option characteristics. The upper and lower price limits start to price the option correctly to two decimal figures at the 200-grid-density-mark in case of the simple calculation method, while the more efficient method reaches this target already at the 125-grid-density-mark and is about 65% faster. Further improvements in the precision are very costly in case of the simple method. The more efficient method reduces the cost immensely, but the calculation time of just one single precise price is still significant.

Improving the grid yields big precision/time gains, and the more efficient calculation method used here is by far not optimal. However, with increasing sophistication of the construction of the grid, either a good notion of how to space the knots before running the program is needed, or the code has to be extended to include a method to construct the grid in a more efficient way without additional user input.

<sup>4</sup> Calculation executed on i7-3517U CPU 2.4 GHz, 6.00 GB RAM

## 5.2 Performance of the Black approximation

Given that the Black approximation is a far simpler and faster, but not a precise method to calculate the value of an option in the presence of a discrete dividend payment, it is important to know when it performs acceptably well and when the calculated prices deviate from the true prices by a non-negligible amount. Based on the simulation results shown in Figure 7, the most important observation is that the Black approximation always falls short of the true price. As can be expected, the more likely the non-dividend paying asset is to be the determining factor at maturity, the less the percentage difference as the dividend payment loses importance. The likelihood is driven by the initial prices of the two assets in combination with their respective volatilities and the size of the dividend payment. General rules of thumb are difficult to derive due to this interdependency, but one should be more and more careful the less clear it is which asset will rank on top at maturity. Also, if the dividend paying asset is more likely to determine the payoff, the higher the dividend, the less exact the Black approximation is. Finally, one will encounter less proportionate mispricings for higher levels of correlation and interest rates.



**Figure 7:** Additional value of the true price in comparison to the price based on Black's approximation for different option characteristics (if not stated differently:  $A = 75$ ,  $B = 55$ ,  $K = 60$ ,  $D = 5$ ,  $\rho = 0.7$ ,  $\sigma_A = 0.2$ ,  $\sigma_B = 0.3$ ,  $r = 0.05$ ,  $T = 1$ ,  $\tau = 0.5$ ,  $t = 0$ )

## 5.3 Other payoff functions for bivariate options

As remarked by Stulz (1982), the bivariate European call-on-max can be replicated as the sum of the two univariate European calls minus the bivariate European call-on-min with the same maturity and strike for all options. In the case of the bivariate call-on-max

where one underlying pays a discrete dividend, this reasoning still holds true. The only adaption is that both the price of the univariate call on the dividend-paying underlying and the price of the bivariate call-on-max have to be calculated with the formula of Amaro de Matos et. al. for  $C^{MDF}(A, K, D, t)$  and the formula derived in this work project.

$$Z_{\min}^{\text{Call}}(A, B, K, D, t) = C^{MDF}(A, K, D, t) + C(B, K, t) - Z_{\max}^{\text{Call}}(A, B, K, D, t) \quad (70.)$$

Note that in this way,  $Z_{\min}^{\text{Call}}(A, B, K, D, t)$  can be priced, even though the option price surface during the life of the option is neither strictly convex nor strictly concave.

Similarly, using Stulz's results for bivariate put options yields the following formulas for the bivariate put-on-max and put-on-min:

$$Z_{\max}^{\text{Put}}(A, B, K, D, t) = e^{-r(T-t)}K + Z_{\max}^{\text{Call}}(A, B, K, D, t) - Z_{\max}^{\text{Call}}(A, B, 0, D, t) \quad (71.)$$

$$Z_{\min}^{\text{Put}}(A, B, K, D, t) = e^{-r(T-t)}K + Z_{\min}^{\text{Call}}(A, B, K, D, t) - Z_{\min}^{\text{Call}}(A, B, 0, D, t) \quad (72.)$$

#### 5.4 Special cases of bivariate options with simplified solutions

There are two special cases in which a bivariate call-on-max in the presence of a discrete dividend can be priced far easier than with the formula proposed in this work project. The first arises in a situation in which either the value of asset A exceeds the value of asset B or vice versa and based on the other factors like volatilities and maturity a switch in the ranking is virtually impossible. In the case that A is far bigger than B, the payoff function of the call collapses from  $Z_{\max}(A, B, K, 0) = \max[\max(A, B) - K; 0]$  to the usual one-dimensional call payoff function  $Z_{\text{Call}}(A, K, 0) = \max[A - K; 0]$ . The use of the easier formula proposed by Amaro de Matos et al. (2009) is sufficient in this case, as the bivariate part of the option has no effect on the price.

The second special case consists of bivariate options on underlyings which are not correlated. Note that in the variable transformation which is used to transform the initial PDE into the heat equation, the coefficient of correlation  $\rho$  only appears in the definition of  $x_1$ . Setting  $\rho$  to zero yields symmetric expressions for  $x_1$  and  $x_2$ . Particularly,  $x_1$  does not depend on asset B anymore, but is determined by asset A only.

$$x_1 = \frac{-\theta[2r - \sigma_A^2] - 2\log(A)}{\sqrt{2}\sigma_A} \quad (73.)$$

$$x_2 = \frac{-\theta[\sigma_B^2 - 2r] + 2\log(B)}{\sqrt{2}\sigma_B} \quad (74.)$$

This cancels out the problem of having interdependent limits of integration in formula (69.), which turns the double integral into the bivariate standard normal cumulative distribution function for which tabulated values exist.

### 5.5 Pricing multivariate options where one underlying pays a discrete dividend

Given that the approach introduced by Amaro de Matos et al. can be extended from the one asset case to the two assets case, the question arises if this is also true for higher dimensions. Without prove, but based on the similarity between the problem structure for the one, two and three assets case, it should be possible to derive an upper and lower bound for the price of a call-on-max option for any number of underlyings. To see this, note that for any number of assets the derivation of the PDE does not change in a fundamental way: based on the multivariate Itô formula, further terms including first, second and mixed derivatives of the newly introduced assets appear and are similar to the already existing terms. The search for an appropriate change of variables also follows a clear pattern. First, write down symbolically the change of variables that shall be introduced (i.e., as in this work project,  $\Phi(x, \theta)$ ,  $\theta$ ,  $x_1$ ,  $x_2$ ) and replace the derivatives from the PDE with the new derivatives for the new variables. Second, group the terms with regard to the  $\Phi(x, \theta)$ -derivatives and note that the continuation of the relationships following the behavior in table Table 3 can be assumed for any number of assets:

if $\Phi(x_1, \theta)$	$x_1(A, t)$
if $\Phi(x_1, x_2, \theta)$	$x_1(A, B, t), x_2(B, t)$
if $\Phi(x_1, x_2, x_3, \theta)$	$x_1(A, B, C, t), x_2(B, C, t), x_3(C, t)$

**Table 3: Dependence structure of the change of variables for varying number of underlyings**

This means that many of the pure second  $\Phi(x, \theta)$ -derivative terms are equal to zero. In the third step, find the solution to the PDE with the new variables while defining all pure second  $\Phi(x, \theta)$ -derivative factors to be the same constant  $\kappa$ . In the one, two and three assets case, this leads to the following definitions of  $x_1$ ,  $x_2$  and  $x_3$ :

$$x_1^{1D} = \frac{\sqrt{2}\sqrt{\kappa^{1D}}(\log(A) + K_1^{1D})}{\sigma_A} \quad (75.)$$

$$x_1^{2D} = \frac{\sqrt{2}\sqrt{\kappa^{2D}}(\log(A) + \log(B)K_1^{2D} + K_2^{2D})}{\sqrt{\sigma_A^2 + \sigma_B^2(K_1^{2D})^2 + 2\rho\sigma_A\sigma_B K_1^{2D}}} \quad (76.)$$

$$x_2^{2D} = \frac{\sqrt{2}\sqrt{\kappa^{2D}}(\log(A) + K_3^{2D})}{\sigma_A} \quad (77.)$$

$$x_1^{3D} = \left[ \sqrt{2}\sqrt{\kappa^{3D}}(\log(A) + \log(B)K_1^{3D} + \log(C)K_2^{3D} + K_3^{3D}) \right] \\ * \left[ \sigma_A^2 + \sigma_B^2(K_1^{3D})^2 + \sigma_C^2(K_2^{3D})^2 + 2\rho_{AB}\sigma_A\sigma_B K_1^{3D} + 2\rho_{AC}\sigma_A\sigma_C K_2^{3D} \right. \\ \left. + 2\rho_{BC}\sigma_B\sigma_C K_1^{3D}K_2^{3D} \right]^{-0.5} \quad (78.)$$

$$x_2^{3D} = \frac{\sqrt{2}\sqrt{\kappa^{3D}}(\log(A) + \log(B)K_4^{3D} + K_5^{3D})}{\sqrt{\sigma_A^2 + \sigma_B^2(K_4^{3D})^2 + 2\rho\sigma_A\sigma_B K_4^{3D}}} \quad (79.)$$

$$x_3^{3D} = \frac{\sqrt{2}\sqrt{\kappa^{3D}}(\log(A) + K_6^{3D})}{\sigma_A} \quad (80.)$$

The sequences of  $x_1$ ,  $x_2$  and  $x_3$  follow a pattern in which the variables of the former dimensionality build the variables of the next dimensionality, expanded by another

variable which resembles the already existing ones. Besides  $\kappa$ , starting for the two asset case, there are other constants  $K_i^{1D}$  which can subsequently be found by setting the factors of the first and mixed  $\Phi(x, \theta)$ -derivatives equal to zero.  $\kappa$  can be chosen freely to either make the formulas easier to read or facilitate the calculations. Amaro de Matos et al. implicitly choose  $\kappa^{1D} = \frac{1}{2}\sigma_A^2$ , while  $\kappa^{2D} = 1$  in this work project.

The proof of convexity is straightforward once the PDE is in the form of the heat equation and the construction of surfaces and integration to receive the bounds should be arduous but doable. However, given that the computational effort rises more than proportionally with the number of assets involved and keeping in mind the limitations mentioned in the next section, the derivation of the formulas should be irrelevant from the standpoint of a practitioner.

## 6. Limitations of the results

It is clear that the approach taken in this work project does not price traded bivariate options correctly. The model is based on the Black & Scholes economy, in which the risk free interest rate, volatilities of the asset returns and the correlation between the assets are constant over the life of the option. Furthermore, it is assumed that the asset prices follow a lognormal process, which can easily be rejected by market data for single stocks. The model is closer to reality than the Stulz pricing formula because a single discrete dividend payment can now be taken into account. However, even this is only possible due to quite escapist assumptions, namely that the time and amount of the dividend payment is precisely known before it is actually paid. Also, both assets might pay a dividend, or one asset might pay more than one dividend. Even though the model surpasses the Black approximation in precision, depending on the option characteristics the approximation still works reasonably well and is always far easier and faster to apply. Finally, due to the need for numerical integration, this approach loses its appeal of providing quasi-analytic formulas for pricing a bivariate option with a discrete dividend payment and is quite time-intensive.

## 7. Conclusion

In this work project the approach introduced by Amaro de Matos et al. of pricing a European option on an underlying which pays a discrete dividend is extended to the bivariate case. While the steps to reach formulas for the upper and lower price of the option follow the same pattern, the correlation between the assets gives rise to interdependent limits of integration and deprives the formulas of being based on tabulated values. Simulations of the precise bivariate option prices show that prices calculated with Black's approximations understate the true prices by an amount dependent on the option characteristics. Due to the structure of the partial differential

equation and change of variables it should theoretically be possible to price options with any number of underlyings in the presence of one dividend payment using this approach.

## Annex

Transformation of the initial PDE into the heat equation

$$\theta = T - t \quad (81.)$$

$$x_1 = \frac{-\theta[2r\sigma_B + \rho\sigma_A\sigma_B^2 - 2r\rho\sigma_A - \sigma_A^2\sigma_B] - 2\sigma_B\log(A) + 2\rho\sigma_A\log(B)}{\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \quad (82.)$$

$$x_2 = \frac{-\theta(\sigma_B^2 - 2r) + 2\log(B)}{\sqrt{2}\sigma_B} \quad (83.)$$

$$\Phi(x, \theta) = e^{r(T-t)}Z(A, B, t) \quad (84.)$$

$$\frac{\partial Z}{\partial t} = rZ - \frac{\partial Z}{\partial A}Ar - \frac{\partial Z}{\partial B}Br - \frac{1}{2}\frac{\partial^2 Z}{\partial A^2}\sigma_A^2A^2 - \frac{\partial^2 Z}{\partial A\partial B}\rho\sigma_A\sigma_BAB - \frac{1}{2}\frac{\partial^2 Z}{\partial B^2}\sigma_B^2B^2 \quad (85.)$$

$$\frac{\partial Z}{\partial t} = e^{-r\theta} \left[ r\Phi + \frac{\partial \Phi}{\partial x_1} \frac{2r\sigma_B + \rho\sigma_A\sigma_B^2 - 2r\rho\sigma_A - \sigma_A^2\sigma_B}{\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} + \frac{\partial \Phi}{\partial x_2} \frac{\sigma_B^2 - 2r}{\sqrt{2}\sigma_B} - \frac{\partial \Phi}{\partial \theta} \right] \quad (86.)$$

$$\frac{\partial Z}{\partial A} = e^{-r\theta} \left[ -\frac{\partial \Phi}{\partial x_1} \frac{2\sigma_B}{A\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \right] \quad (87.)$$

$$\frac{\partial Z}{\partial B} = e^{-r\theta} \left[ \frac{\partial \Phi}{\partial x_1} \frac{2\rho\sigma_A}{B\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} + \frac{\partial \Phi}{\partial x_2} \frac{2}{B\sqrt{2}\sigma_B} \right] \quad (88.)$$

$$\frac{\partial^2 Z}{\partial A^2} = e^{-r\theta} \left[ \frac{\partial^2 \Phi}{\partial x_1^2} \left( \frac{2\sigma_B}{A\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \right)^2 + \frac{\partial \Phi}{\partial x_1} \frac{2\sigma_B}{A^2\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \right] \quad (89.)$$

$$\frac{\partial^2 Z}{\partial A\partial B} = e^{-r\theta} \left[ -\frac{\partial^2 \Phi}{\partial x_1^2} \frac{2\sigma_B}{A\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \frac{2\rho\sigma_A}{B\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} - \frac{\partial^2 \Phi}{\partial x_1\partial x_2} \frac{2\sigma_B}{A\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \frac{2}{B\sqrt{2}\sigma_B} \right] \quad (90.)$$

$$\frac{\partial^2 Z}{\partial B^2} = e^{-r\theta} \left[ \frac{\partial^2 \Phi}{\partial x_1^2} \left( \frac{2\rho\sigma_A}{B\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \right)^2 - \frac{\partial \Phi}{\partial x_1} \frac{2\rho\sigma_A}{B^2\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} + 2\frac{\partial^2 \Phi}{\partial x_1\partial x_2} \frac{2\rho}{B^2\sigma_B^2\sqrt{1-\rho^2}} + \frac{\partial^2 \Phi}{\partial x_2^2} \left( \frac{2}{B\sqrt{2}\sigma_B} \right)^2 - \frac{\partial \Phi}{\partial x_2} \frac{2}{B^2\sqrt{2}\sigma_B} \right] \quad (91.)$$

Inserting (86.)-(91.) into the PDE (85.) results in the heat equation

$$\frac{\partial \Phi}{\partial \theta} = \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} \quad (92.)$$

Transformation of the boundary condition

$$\Phi(x, 0) = Z(A, B, T) = \max[\max(A, B) - K; 0] \quad (93.)$$

$$(x_2 | \theta = 0) = \frac{2\log(B)}{\sqrt{2}\sigma_B} \rightarrow (B | t = T) = \exp\left(\frac{x_2\sqrt{2}\sigma_B}{2}\right) \quad (94.)$$

$$(x_1 | \theta = 0) = \frac{2\rho\sigma_A\log(B) - 2\sigma_B\log(A)}{\sqrt{2}\sigma_A\sigma_B\sqrt{1-\rho^2}} \rightarrow (A | t = T) \quad (95.)$$

$$= \exp\left(\left(\rho x_2 - \sqrt{1-\rho^2}x_1\right)\frac{\sigma_A}{\sqrt{2}}\right)$$

$$\Phi(x, 0) = \max\left[\max\left[\exp\left(\left(\rho x_2 - \sqrt{1-\rho^2}x_1\right)\frac{\sigma_A}{\sqrt{2}}\right); \exp\left(\frac{x_2\sqrt{2}\sigma_B}{2}\right)\right] - K; 0\right] \quad (96.)$$

Partial derivative of  $Z_{\max}^{\text{Call}}(A, B, K, t)$

$$\frac{\partial Z_{\max}^{\text{Call}}(A, B, K, t)}{\partial A} \quad (97.)$$

$$\begin{aligned} &= N_2 \left( -d_{-}^{B/A}, d_{+}^A, \rho_{BK,A} \right) + \frac{\exp \left( -\frac{1}{2} (d_{-}^{B/A})^2 \right)}{\sigma_{A/B} \sqrt{2\pi} \sqrt{T-t}} N_1 \left( \frac{d_{+}^A + \rho_{BK,A} d_{-}^{B/A}}{\sqrt{1 - \rho_{BK,A}^2}} \right) \\ &+ \frac{\exp \left( -\frac{1}{2} (d_{+}^A)^2 \right)}{\sigma_A \sqrt{2\pi} \sqrt{T-t}} N_1 \left( \frac{-d_{-}^{B/A} - \rho_{BK,A} d_{+}^A}{\sqrt{1 - \rho_{BK,A}^2}} \right) \\ &- \frac{B \exp \left( -\frac{1}{2} (d_{-}^{A/B})^2 \right)}{A \sigma_{A/B} \sqrt{2\pi} \sqrt{T-t}} N_1 \left( \frac{d_{+}^B + \rho_{AK,B} d_{-}^{A/B}}{\sqrt{1 - \rho_{AK,B}^2}} \right) \\ &- \frac{K \exp(-r(T-t)) \exp \left( -\frac{1}{2} (d_{-}^A)^2 \right)}{A \sigma_A \sqrt{2\pi} \sqrt{T-t}} N_1 \left( \frac{-d_{-}^B + \rho d_{-}^A}{\sqrt{1 - \rho^2}} \right) \end{aligned}$$



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